The Logic in Logicism

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I would like to dedicate this paper to the memory of George Boolos, to whose conversation and writings my thinking on Frege and logicism, and this paper in particular, owe so much.

RÉSUMÉ : Le logicisme de Frege engage à deux thèses : (1) les vérités de l'arithmétique sont ipso facto *vérités de logique; (2) les nombres naturels sont des objets. Dans cet article je pose la question : quelle conception de la logique est-elle requise pour étayer ces thèses? Je soutiens qu'il existe une conception appropriée et naturelle de la logique, en vertu de laquelle le principe de Hume est une vérité logique. Le principe de Hume, qui dit que le nombre de* Fs *est le nombre de* Gs *ssi les concepts* F *et* G *sont équinumériques, est la pièce maîtresse de l'argumentation néologiciste en faveur de (1) et (2). Je défends cette position contre deux objections: a) le principe de Hume ne peut pas à la fois être d'ordre logique, comme le requiert (1), et avoir la portée ontologique requise par (2); et b) l'usage logiciste du principe de Hume à l'appui de (2) se ramène à une preuve ontologique d'un genre plus large, qui est non valide.*

1.

Arithmetical logicism is the claim that the truths of arithmetic are consequences of truths of logic, and are thus themselves logical truths. Famously, Frege was a logicist; he set out the principles of his logicist program in his *Foundations of Arithmetic* (*Grundlagen der Arithmetik*) and sought to carry them out formally in the *Basic Laws of Arithmetic* (*Grundgesetze der Arithmetik*). As is well known, Russell showed the

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axiom system Frege used in the *Basic Laws* to be inconsistent. Had Frege carried out this plan in the way he had laid out in the *Foundations*, however, his system would not have suffered from the Russellian contradictions.[1] In the *Basic Laws* he employs an axiom—the fifth basic law, which is central to the derivation of the contradictions—whereas in the *Foundations* he describes a weaker principle, called Hume's principle:

> Hume's principle (N): the number of Fs = the number of Gs iff Fs and Gs can be paired off exactly (Fs and Gs can be 1-1 correlated)

Being weaker than the fifth basic law, Hume's principle does not lead to a contradiction, but it is strong enough to give us arithmetic.

Another feature of Frege's *Foundations* is its Platonism, as rooted in the context principle. While the exact nature of the context principle is a matter of debate, the Platonist conclusion is that reference is conferred on an expression by its use as a singular term in a true sentence. Numerical expressions in particular, as introduced by Hume's principle, function as singular terms in true sentences, and so we are justified in claiming the existence of numbers as the referents of those expressions.

Both features of the Fregean program—the logicism and the Platonism—have found support in Crispin Wright's *Frege's Conception of Numbers as Objects*, while the Platonism has received a more general application and justification in Bob Hale's *Abstract Objects*. In this paper I shall consider that notion of logic which is appropriate to Platonist logicism and look at how it may be employed in the defence of this position.

2.

A reasonably precise conception of logic is necessarily an important issue for a logicist. It is not entirely clear, however, what logic is supposed to be—i.e., what principles should delimit the scope of logic.[2] In order to clarify the question, and in order to reach a principled notion of logic, I will offer two characterizations: narrow logic and broad logic.

Narrow logic concerns itself with rules, principles, axioms, and the like which turn upon the employment of certain typically "logical" concepts, essentially truth-functions—not, and, or, if . . . then—and quantifiers—all, some, any, and so on. It is at the level of narrow logic that discussions of the scope of logic always take place. It is narrow logic one is talking about when one asks, "Should logic include axioms for the use of modal operators or for set-membership?"

Broad logic is informed by the thought that understanding a proposition involves grasping the logically valid inferences in which that proposition is involved. For instance, if one understands "It is raining, but he is singing" one must know that this entails "It is raining." More specifically, understanding a concept involves an appreciation of the pattern of logi-

cally valid inferences that propositions employing that concept are part of in virtue of employing that concept. Understanding "green" and "red," for instance, requires appreciating that from "The apple is red all over," one may infer "The apple is not green." This principle I shall call the Principle of Logical Understanding (PLU). Broad logic says that a satisfactory characterization of logic should account for the logical validity of those inferences identified as such by PLU. Thus logic should make it the case that the inferences

(1a) It is raining but he is singing $\vdash$ it is raining

(1b) The apple is red all over $\vdash$ the apple is not green

are inferences valid by logic alone.

For much of the first part of the century it was believed that narrow logic and broad logic were the same thing. Wittgenstein's *Tractatus* is the best, but not the only possible, exposition of this view. According to the *Tractatus*, the validity of the inferences will be seen once they are suitably analyzed, since the corresponding implications:

(2a) It is raining but he is singing $\rightarrow$ it is raining

(2b) The apple is red all over $\rightarrow$ the apple is not green

will be seen to be tautologies.

There are significant obstacles in the way of such claims; the very act of analysis involves making inferences. But from where do these inferences get their logical explication? Furthermore, such an approach looks feasible in the first case, if one attaches to "but" the truth-table normally given to the logical "and." But this would seem to lose some of the conversational implicature involved in the use of the word "but" rather than "and."

Most importantly, the project of such analysis cannot be carried out in the cases of many inferences, including (1b). For the project to be successful, it must be possible to analyze any proposition into a combination of logical concepts—that is, narrowly logical—and other concepts which themselves play no part in any logical relation. Wittgenstein summed this requirement up by saying that elementary propositions—propositions comprising just such basic non-logical concepts—are logically independent. But, as he himself later realized on trying to find an analysis for the second inference, there simply is not a stock of such logically independent elementary propositions. The upshot is that there are inferences which PLU says are broadly logical but which are not narrowly logical.[3]

I mention this distinction in order to make it clear that logicism says that arithmetic is broadly logical, not narrowly logical. Fundamentally, arithmetic employs only one concept not normally attributed to narrow

logic, but since this concept—the concept of *number* itself—is so important, it is least question-begging to regard it as part of narrow logic. As we shall see, the narrow logic required by logicism is second order. Quine and others would reject the claim that second order logic is truly (narrow) logic. But in the context of my claim that the scope of narrow logic is neither clear nor of great philosophical interest, and the view that the logic in logicism is broad logic, Quine's view is not especially germane. That second-order logic is indeed part of broad logic at least is to be seen by considering that an inference such as:

> From "Katherine of Aragon, Anne Boleyn, Anne of Cleves, and Catherine Parr were all wives of Henry VIII" infer "There is some property which Katherine of Aragon, Anne Boleyn, Anne of Cleves, and Catherine Parr had in common"

is an inference which is logically valid according to PLU.

It is worth noting that every broadly logical inference can be made to look narrowly logical by pulling out a supposedly hidden major premise. The conclusion "The apple is not green" follows, by narrow logic alone, from "The apple is red all over" *plus* the additional premise, "Nothing may be red and green all over at the same time." Such additional or hidden premises are enthymemes. But their existence does not show that broad logic and narrow logic are the same after all; all it shows is that these enthymemes are logical truths—that is, broadly logical truths. Sometimes broadly logical truths are called conceptual truths, because of their connexion via PLU with our understanding of the concepts they employ. It is in this sense that Hume's principle is a conceptual truth.

It might seem possible to regard these enthymemes as non-logical metaphysical truths. This position might allow for a restriction of logic to narrow logic. The logicist must reject this, however, if he or she is to defend logicism against its critics. But the view that all necessity is logical, and hence conceptual, necessity is itself defensible, for it provides an explanation of necessity which otherwise remains mysterious or ontologically uninviting. Because of this, the onus is on those who think that there is such a thing as non-logical necessity to provide both examples and an explanation of its nature. In proposed cases, such as Kripke's essentialism, the necessity involved can be shown to be inherited from a logical necessity.

Let us now return to the logicist claim, which is that all that is essential to arithmetic can be derived from narrow logic plus the broad logical truth, Hume's principle. ("Derived" here means narrowly derived. Indeed, all one need say is that arithmetic is narrowly derived from Hume's principle simpliciter.[4])

(3) Logicism: Hume's principle $\vdash_{\text{narrow logic}}$ arithmetic

That Hume's principle is a broadly logical or conceptual truth can be seen by considering pairs of propositions such as:

(4a) There were as many fates as magi

(4b) The number of fates is the same as the number of magi

(5a) There are as many provinces in Canada as parishes in Dominica

(5b) The number of provinces in Canada is the same as the number of parishes in Dominica

In each pair, the second proposition follows from the first by a valid inference, and vice versa. These are facts about the English language such that acknowledging the validity of such inferences would normally be regarded as a criterion of understanding the key constituent expressions—in this case, the expressions "as many as" and "the number of." (I say "normally" since a philosopher may well deny the validity of what everyone else recognizes to be valid, in which case the onus is very much on that philosopher.) Since inferences of the following form are valid:

(6a) There are as many Fs as Gs $\vdash$ the number of Fs is the number of Gs

(6b) The number of Fs is the number of Gs $\vdash$ there are as many Fs as Gs

it follows from PLU that the corresponding enthymeme—viz., Hume's principle—is a logical truth.

Returning to the logicist claim made at (3), that arithmetic can be derived by narrow logic alone from Hume's principle, let us see how this works in practice. First note that Hume's principle guarantees the existence of numbers. A particular case of the principle is where the concepts F and G are identical:

(7) the number of Fs = the number of Fs iff Fs and Fs can be paired off exactly

Any sufficiently well-defined concept can be 1-1 correlated (exactly paired off) with itself (e.g., by the relation of identity). The notion of 1-1 correlation is itself a purely (narrow) logical notion, which can be seen when 1-1 correlation between F and G is spelt out thus:

(8) $\exists R\{\forall x\,(Fx \rightarrow \exists y(Gx \,\&\, Rxy)) \,\&\, \forall y(Gy \rightarrow \exists x(Fx \,\&\, Rxy))$

$\&\; \forall x\forall y\forall z[Rxy \,\&\, Rzy \rightarrow (Fx \rightarrow (Fz \rightarrow (Gy \rightarrow x{=}z)))$

$\&\; Rxy \,\&\, Rxz \rightarrow (Fx \rightarrow (Gy \rightarrow (Gz \rightarrow y{=}z)))]\}$

It is the existential quantification over relations in this definition of 1-1 correlation which makes the logic in logicism second order. If we put $Gx \equiv Fx$ and substitute x=y for occurrences of Rxy, etc., we have:

(9) $\forall x(Fx \rightarrow \exists y(Fx \,\&\, x{=}y)) \,\&\, \forall y(Fy \rightarrow \exists x(Fx \,\&\, x{=}y))$
$\&\, \forall x \forall y \forall z[x{=}y \,\&\, z{=}y \rightarrow (Fx \rightarrow (Fz \rightarrow (Fy \rightarrow x{=}z)))$
$\&\, x{=}y \,\&\, x{=}z \rightarrow (Fx \rightarrow (Fy \rightarrow (Fz \rightarrow y{=}z)))]$

which is a logical truth for any well-defined concept F. Thus, from (7) we have:

(10) the number of Fs = the number of Fs

from which it follows that something is the number of Fs. This is the point at which the claim deriving from the context principle applies: reference is conferred on an expression by its use as a singular term in a true sentence.[5] In the light of the rule of existential generalization (from Fa infer $\exists x\, Fx$), this claim is equivalent to taking an objectual reading of all first-order quantifiers.

It may be asked why we cannot get (10) immediately from the law of identity $\forall x(x{=}x)$. My view is that this law can only be applied when the objects in question have been shown, or are at least assumed, to exist; i.e., if "**c**" is an empty singular term, **c=c** cannot be true. Therefore, it is illegitimate to infer from $\forall x(x{=}x)$ that **d=d** unless we have a guarantee that **d** is not empty. In formal systems we may provide a stock of basic singular terms, but we do so on the understanding that any satisfactory interpretation will provide each one with a corresponding object. In the case of complex or derived singular terms, the matter is not so simple. I may wish, for example, to formalize various facts about the people who came to dinner last night. I may start by providing basic names to stand for those people: **a** for Aristotle, **b** for Buridan, **c** for Carnap, **d** for DeBeauvoir, and so on. One of the sort of facts I want to formalize is about the spouses of my guests, spouses who may or may not have been among the guests. To do so I will introduce a singular term forming operator **s**() standing for "the spouse of . . ." Thus, **s(a)**, **s(b)**, **s(c)** will all be expressions of my system. It will be valid to employ the law of identity to yield **a=a**, and hence $\exists x$ x=**a**, but this is because the system is built upon the intention that the basic names stand for objects. By contrast, there is no such assumption regarding the derived terms **s(a)**, **s(b)**, and **s(c)**. Before we can employ them, we must be sure that they stand for something. To demonstrate this, it will be sufficient to show that the expression in question may figure in a true statement of identity. Corresponding to these requirements, expressions of the form **s**(x) will be introduced via the following rules:

(11a) "**s**(x)" will be a singular term of the language where x is a singular term and some statement of the form $\exists!y(Sxy)$ is true.

(11b) Propositions of the form $\phi(\mathbf{s}(x))$ have their truth conditions determined by the following biconditional: $\phi(\mathbf{s}(x)) \leftrightarrow \exists!y(Sxy \,\&\, \phi(y))$. (In particular $z=\mathbf{s}(x)) \leftrightarrow \exists!y(Sxy \,\&\, z=y)$.)

Our system may show that the derived term **s**(**a**) is a genuine singular term. For instance I may discover at my party that everyone who is happy has a (unique) spouse and that Aristotle is happy. If H() means ' . . . is happy' and Sxy means 'y is espoused to x' the facts can be formalized:

(12a) $\forall x\,(Hx \rightarrow \exists!y(Sxy))$

(12b) **Ha**

from which it follows that $\exists!y(Say)$. (11a) legitimates the use of **s**(**a**) as a singular term, while from (11b) it follows that $\$y(y=\mathbf{s}(\mathbf{a}))$, $\mathbf{s}(\mathbf{a})=\mathbf{s}(\mathbf{a})$and so on.[6]

Hume's principle gives us a number for each concept, and tells us that the same number attaches to different concepts so long as they can be 1-1 correlated. We are in a position to introduce the names of individual numbers: five is the number of fingers (including thumbs) on my left hand; equally, I could have said that five is the number of toes on my right foot, the fingers and toes being 1-1 correlated. Indeed, it is this fact—that 1-1 correlation yields equivalence classes among concepts such that, if x is the number of one of the concepts within a class, it is the number of all of them—which allows us to use fingers, notches on a stick, a sequence of numerals, or sounds as equivalent ways of counting. In case the use of these empirical-looking concepts might suggest that a whiff of the empirical is being introduced into logicism, consider that in defining the number 0 we can find a concept which we can guarantee has no instances—the concept of non-self-identity. Using this as both F and G in Hume's principle, we generate the existence of a number: zero. If we let Fx in (9) be $x\neq x$, then we have as the right-hand side of Hume's principle the logical truth:

(13) $\forall x\,(x\neq x \rightarrow \exists y(x\neq x \,\&\, x=y)) \,\&\, \forall y(y\neq y \rightarrow \exists x(Fx \,\&\, x=y))$

$\&\, \forall x\forall y\forall z[x=y \,\&\, z=y \rightarrow (x\neq x \rightarrow (z\neq z \rightarrow (y\neq y \rightarrow x=z)))$

$\&\, x=y \,\&\, x=z \rightarrow (x\neq x \rightarrow (y\neq y \rightarrow (z\neq z \rightarrow y=z)))]$

To which the corresponding left-hand side is: $\#(x\neq x) = \#(x\neq x)$, where #F means the number of things which are F. "0" can be regarded as shorthand for $\#(x\neq x)$. Now we can find a concept which has precisely one

instance—the concept of being identical to 0. The number of this concept is 1 ($1 = \#(x=(\#(x\neq x)))$). Two is the number of the concept of being identical to 0 or to 1 ($2 = \#(x=0 \vee x=1)$), and so on.

This does not yet look like arithmetic; we have not yet said anything about the properties of numbers and relations among them.[7] But the foregoing remarks suggest that framing the concept of a successor to a number should be reasonably straightforward. Using this notion, we can construct the ancestral of the successor relation, which holds between two numbers if they are indirectly related by a series of successor relations. Something is then a natural number if it is either zero or related to zero by the ancestral of the successor relation. From this definition of a natural number the principle of mathematical induction flows directly—as do, with a little work, the remaining Peano axioms for arithmetic.

This formal aspect of the program is not in doubt. What is in doubt is whether logicism is genuinely logical or whether it delivers genuine objects. And since the only non-narrowly logical item in neo-logicism is Hume's principle, and because the argument for numbers as objects also stems from Hume's principle, it is upon that principle that scorn is heaped by logicism's detractors.

I want to consider two sources of criticism. The first is that Hume's principle is true, but does not have the ontological significance I have given it; this is the reductionist approach. The second is that the principle is not even true; this is what Wright calls the rejectionist approach.

3.

The first approach maintains that the very truth of the principle shows how to "reduce away" numbers. It shows how we do not have to take numerical expressions seriously from an ontological point of view. They may behave as syntactic singular terms, but they do not refer to anything. The reductionist will claim that the logicist has been misled by the superficial syntactic features of our numerical language; the right-hand side of Hume's principle shows what is really there.

The reductionist is thus claiming that there is an asymmetry in Hume's principle: the left-hand side is misleading, the right-hand side is not. But one could equally argue in the opposite direction, that the right-hand side is misleading, not the left. If there is an asymmetry, which is the right way round? The reductionist must have some principled reason for chosing one side over the other as veracious.[8] One such reason would be the complaint that if numerical expressions really do stand for objects, then we could not have the appropriate causal connections with those objects that would be necessary in order to refer to them or to know facts about them.

There are good, general reasons for thinking that this line against abstract objects does not work. Bob Hale argues that the causal theories of knowledge and reference are mutually supporting, rather than both

standing upon some independent basis.[9] Additionally, in the numerical case, the motivation for a causal theory of reference (as opposed, for instance, to a descriptive theory) is lacking. While Kripke shows it is plausible that we can use a name of a person to refer to that person even when we are apprised of no discriminating facts about him or her, it seems absurd to suggest that someone who was genuinely thinking about the number nine could not tell us whether nine came before or after ten in the sequence of natural numbers.

But I think the reductionist may have another motive, one that he or she may share with the rejectionist. It is that logic must be ontologically neutral. The truths of logic should not commit us to the existence of any object. Boolos, for instance, says that any object may not have existed. But if there are logical objects, they would have to exist. Boolos writes:

> We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. After all, logical truth is just *truth no matter what things we may be talking about and no matter what our (non-logical) words mean.* Since there might be fewer than two items that we happen to be talking about, we cannot take even $\exists x \exists y\, x \neq y$ to be valid.[10]

Boolos raises two powerful and related intuitions about the nature of logic and modality: that logic is ontologically neutral, and that, however many entities there actually are, there may have been none at all. Both intuitions are violated by my argument, for the logicist claims to have shown that logic is committed to the existence of infinitely many objects and that they exist necessarily. But the logicist should not be embarrassed by this. After all, that these things are the case is precisely what the logicist is trying to show. The point of logicism is to show that logic entails the existence of infinitely many necessary objects. Clearly, Boolos and the logicist have different intuitions about the nature of logic and its ontological commitment. The question is, then, is there any good ground for preferring Boolos's intuition to that of the logicist?

The italicized portion of the quotation above suggests an argument: that logic simply is not about anything, and therefore cannot be committed to anything. I suggest that this is to conflate reference with ontological commitment. These two things need to be carefully distinguished: on the one hand, what a proposition is about, what it refers to, and on the other, the ontological commitment, what has to exist for the proposition to be true. For while singular reference carries with it ontological commitment, the reverse does not hold; that is, not all ontological commitment is displayed in singular reference. What we say may commit us to the existence of certain things without it being the case that we are talking about them. For instance, if I say "The queen is a grandmother," I am talking about the queen, not about any of her grandchildren. I am nonetheless committed

to the existence of at least one grandchild (at some time or other). And so, while it is the case that we might talk about fewer than two items—indeed, we might talk about none at all—it does not follow that our talk is not ontologically committed. Similarly, to argue against ontological commitment in logic on the ground that logic is "topic neutral" would be to confuse the "topic" of a proposition, i.e., what it is about, and its ontological commitments.

Nonetheless, the reductionist may be able to strengthen Boolos's argument. The reductionist has already maintained that language has a misleading portion and a perspicuous portion. It is precisely in the misleading portion that reference (or explicit existential quantification) and ontological commitment come apart. But in the perspicuous portion it is quite legitimate to conflate the two. (For instance, saying "The queen is a grandmother" in the perspicuous portion—showing the "logical form"—will reveal explicit reference to or quantification over grandchildren.) The reductionist may further maintain that when the axioms and principles of logic are expressed in terms of the perspicuous portion (or some suitable extension or formalization of it), we can see that logic has no ontological commitment. Of course, this is a fact which, in some sense, has to be a logical discovery. Is it one we have, indeed, discovered? It might look that way if one concentrated on systems of formal logic. But that is to focus on narrow logic, and the logicist's claim is not that narrow logic is committed to numbers but that broad logic is. And there just does not seem to be any way of guaranteeing that broad logic is not ontologically committed.

Furthermore, the postulation of a portion of language which displays its ontological commitment perspicuously, and to which the misleading sentences of the language may be reduced, is itself open to question. Were there such a perspicuous portion, it would have to comprise a stock of basic or elementary propositions—expressible in the perspicuous portion and being compounded appropriately—which provide equivalences for any meaningful sentence of the language. Further, the equivalences must be logically independent of one another; if two distinct elementary propositions could be logically equivalent, then each would have the ontological commitment of the other. On the hypothesis that they are genuinely distinct, however, their "face-value" ontological commitments will differ. Thus, the ontological perspicuity of elementary propositions is not compatible with their being logically related. As I have already remarked, this claim about the existence of logically independent elementary propositions is false.

4.

I have maintained that inferences of the form embodied in Hume's principle are valid and must be recognized as such by competent users of

English. Accepting such inferences is part of what is involved in a mastery of numerical concepts. In the light of PLU, I have argued that Hume's principle is therefore a logical truth—which is precisely what the rejectionist rejects. Is it possible to deny the truth of Hume's principle, yet accept the intuition that it plays a central role in fixing the concept of number? There is one model for precisely this. According to this model, the meaning of theoretical terms in science is taken to be fixed by the theory within which they occur. Yet, this is independent of the truth of the theory or of the theoretical terms referring to anything. Thus, we can understand "phlogiston" because of the role the concept played in a theory that turned out to be false.

Is this a satisfactory approach to the meaning of theoretical terms? I think not. To begin with, it is unclear which parts of which theories we are supposed to regard as fixing the meaning of well-established theoretical terms. Which theory determines the concept "electron"? Is it current theory? And if it is, is it current accepted theory or the latest speculative theory? Or is it early theory responsible for introducing and developing the term—the theories of Stoney and Bohr? Since these theories and intervening ones are all different, it would appear that if they fix any concept of electron at all, they all fix different concepts. But this cannot be right; discourse and continuity in science depend on scientists sharing concepts and not on changing them. What makes these various theories all theories employing the same concept is that their development sought to maintain the extension of the concept from their predecessors. In this sense, insofar as theories fix concepts, they fix extensions, not intensions. Furthermore, it would be better to say that subsequent theories do not so much fix as fix upon extensions; they do so by inheriting the explanatory or causal role of theoretical entity or quantity in question from the preceding theories. So the meaning of a theoretical term is not its role in a theory simpliciter but is a matter of its role in explaining certain phenomena, enabling the extension of the term to be maintained among theories, all of which embody the same explanatory role.

By contrast, it does not seem any part of the role of Hume's principle to fix, let alone fix upon, extensions. Rather, its role is to introduce a concept in the sense of providing criteria for the correct use of the concept in a way that is constitutive of the concept. Any replacement of Hume's principle by a different but similar principle could not be regarded as a change in a theory which was nonetheless still about numbers. Any such alternative simply would not and could not have the same extension. Nor is Hume's principle a theory which proposes an explanatory role for numbers. There are no phenomena which are supposed to be explained by the existence of numbers. It is this fact—that Hume's principle is constitutive of the concept of number in the same way that verbal definitions are con-

stitutive of concepts, and theories are not—which means that we have no choice but to regard Hume's principle as true.

A distinct ground for supposing that Hume's principle cannot be a conceptual truth is that it appears to introduce a concept in a way that guarantees there will be things falling under the concept. Given that Kant expressly rejected the possibility of doing this in his critique of ontological proofs of God's existence, it might seem that logicism is condoning a form of ontological proof just as invalid as that criticized by Kant.

The currently accepted response to ontological proofs is that they mistakenly take "exists" to be a first-level predicate and not a second-level predicate (a quantifier). Clearly, the logicist is not making any mistake of this sort; the logicist is simply claiming that numerical expressions are genuine singular terms appearing in true sentences to which the rule of existential generalization may be applied in the normal way.

There may be other fallacies involved in ontological proofs, though, and perhaps the logicist has committed one of these. Let us therefore put the issue of "exists" not being a predicate on one side and see what further problems may arise. A simple ontological proof might run like this: let Ex mean x exists, and let Px mean x possesses the perfections and other attributes appropriate to God. We want God to be such that he satisfies the conjunction Ex & Px. There may be theological or other grounds for thinking that, at most, one being, if any, satisfies Px. We can thus define God as ιx(Ex & Px) without fear that the uniqueness condition of definite descriptions will be violated. We can be sure that the existence condition will not be violated either, since existence (Ex) is part of the definition. To see this, consider the predicate Ex. If it is to confer existence on anything, it will need to be related to the existential quantifier, and so the following will at least have to be the case: $Ex \rightarrow \exists y(y=x)$. Let **g** be shorthand for God—i.e., $\mathbf{g}=\iota x$(Ex & Px). It follows from the definition of the definite description operator that E**g**, and using our rule for E, $\exists y(y=\mathbf{g})$. This would appear to show that God exists, and so any proposal that nothing could satisfy the existence requirement of the definite description would lead to a contradiction.

This argument can be shown to be invalid, but its invalidity does not depend on any untoward feature of the existence predicate, for the only feature of existence we have used is $Ex \rightarrow \exists y(y=x)$. But this is true for any predicate whatsoever. The fault lies with the more general invalidity of assuming that one may use a definite description—or a name standing for one—as a singular term in demonstrating that it has a reference. It is precisely its use as a singular term in true sentences that confers reference. For this reason, use as a singular term does not come cheaply. Before any derived singular term can be used as such, it must be demonstrated that it does have a legitimate use in true sentences. In particular, the singular

term in question must be able to appear in true canonical statements of identity.

In introducing numerical expressions, I was explicit in avoiding this sort of fallacy. I was careful to ensure that what gave us the right to regard such expressions as non-empty is that Hume's principle entails the truth of identity statements containing them. Therefore, the logicist argument could not be a species of this sort of invalid ontological proof.

5.

The question may then arise: while logicism does not employ an Anselm-style ontological proof, could not the new sort it does use be employed in the proof of the existence of things other than numbers—gods, for instance? Let us bring Hume's principle to the fore again. (In what follows I shall refer to Hume's principle as "(N)" in comparison with (G), introduced by Hartry Field.[11]

According to the logicist,

(N) The number of Fs is identical to the number of Gs iff there are exactly as many Fs as Gs

is a conceptual and hence necessary truth. (N) constitutes, in part, the concept of number. When F has a well-defined extension, it is a necessary truth that there are exactly as many Fs as Fs. It follows that, necessarily, the number of Fs is identical to the number of Fs. This identity cannot be true without there being something which is the number of Fs.

Field's objection to this line of argument takes the form of a *reductio* by analogy: if this argument for the conceptual necessity of the existence of numbers were valid, then so would an analogous argument whose conclusion is that creative gods exist. Extensions of the argument could be taken to demonstrate the necessary existence of a variety of entities such as gods whose existence we are inclined to think of as at best contingent, if indeed they exist at all. If Field's analogy is faithful to the Platonist/logicist's argument, clearly the latter would amount to an ontological argument of dubious validity.

In place of (N), Field introduces

(G) The god that created x is identical to the god that created y iff x and y are spatio-temporally related.

Field argues that, if Wright and Hale's arguments regarding (N) were correct, analogous arguments applied to (G) would allow us to infer the existence of a god. (G) yields "the god that created x is identical to the god that created x iff x is spatio-temporally related to itself," from which we have, "There is a god that created x iff x is spatio-temporally related

to itself." Since any item is spatio-temporally related to itself, it follows that there is a god that created x, where x is a spatio-temporal item.

Field rejects this conclusion. While (G) may have something to do with explicating the concept "god," he argues, (G) cannot be a conceptual truth. Rather, the conceptual truth, if any, is: "If gods exist then (G)"; similarly, the conceptual truth in the case of numbers is not (N) but "If numbers exist then (N)." But Field's analogy is not true to (N) and the argument for the existence of numbers that flows from it. That argument can be divided into two parts: an ontological claim and an analytical claim. The ontological claim can be explained as follows. There are conceptual truths of the form:

(I) $\mathbf{s}(\mathbf{a}) = \mathbf{s}(\mathbf{b})$ if and only if $R(\mathbf{ab})$

where "**s**()" is a singular term-forming operator taking as arguments (e.g., "**a**" and "**b**" above) expressions standing for a certain sort of entity—concepts in (N), lines in Frege's example of directions—and where R is an equivalence relation among those entities. For certain relations R, (I) can be regarded as introducing a class of singular terms, those of the form **s**(x). Since (I) also gives such expressions a use in true sentences, reference is also conferred upon them.[12]

This view is quite separate from the analytical claim that, in any particular case of (I), such as (N), the singular terms thereby introduced equate to a certain class of expressions already employed in English or any other natural language (e.g., the numerical expressions of everyday use). This is a task for conceptual analysis. That is, if (N) is a legitimate way of introducing a class of singular terms which thereby refer to a certain sort of entity, it is a further, conceptual question whether those singular terms are indeed the numerical expressions of everyday speech.

While the ontological and the analytical claims are distinct, the truth of both is required for the logicist argument for the existence of numbers as abstract objects. Nonetheless, it is the first of the claims which is the primary bone of contention here, for it is the structure of (I) that Field exploits in his counter-example.

Let us focus, then, on the class of singular terms introduced by (N). The key concept is the function "the-number-of- . . ." Such a singular term-forming phrase will stand for a function taking concepts as arguments and yielding objects (call them N-objects) as its values, thereby associating an object with each concept. This is the ontological claim about (N). What sort of things are N-objects? This question is answered by looking at the right-hand side of (N), which gives us the identity criteria for N-objects. In particular, the right-hand side tells us how they are associated with concepts: viz., concepts which are equinumerous are associated with the same N-object.

It is a further question whether N-objects are numbers—i.e., whether the analytical claim is true. But this is not at issue here, for Field accepts that if numbers and N-objects do exist, then they are the same things.

Note that even if the analytical claim is true, (N) does not of itself tell us about the concept "number." All (N) deals with are noun phrases of the form "the-number-of-Fs." To introduce the sortal concept "number," we must add: numbers are those objects whose existence is given and identity criteria determined by (N)—i.e., numbers are N-objects. This may be a trivial step, but it is a further step nonetheless.

If (G) were analogous, then what (G) would explain is the function "the-god-which-created . . ."—a function which yields objects (G-objects) for objects as arguments, such that entities which are spatio-temporally related are associated with the same G-object. It would be a mistake to think that (G) could explain the concept "god" simpliciter. As in the case of number, a further step is required to introduce such a concept. It might be: gods are those objects whose existence is given and identity criteria determined by (G)—i.e., gods are just G-objects. Field's argument is that if (N) yields the existence of numbers, then (G) yields the existence of gods, the implication being that the latter is an objectionable conclusion.

But is it? Remember that as far as the ontological claim is concerned, gods are precisely those entities whose identity criteria are given by (G). The concept "god" is understood by knowing that the identity of gods (G-objects) is determined by the spatio-temporal relatedness of the objects with which they are associated. And indeed, there are such objects. The G-object or god associated with the spatio-temporal object x such that the same G-object is associated with any other spatio-temporal object with which x is spatio-temporally related, is the *world* or *universe* to which x belongs. For it is the case that, for each spatio-temporal object, there is a world to which it belongs, and any other entity to which it is spatio-temporally related belongs to the same world.

The constituent words of the phrase "the-god-that-created-x" are irrelevant to the ontological claim. If (G) is to be regarded as a conceptual truth analogous to (N), then that phrase as a whole names an object and has a meaning given by (G), much the same meaning as the phrase, "the world to which x belongs." It is only when we turn to the analytical claim and think of the constituent words "god" and "created" as having independent, antecedently given meanings, that Field's analogy yields its objectionable conclusion. But if we regard "the-god-that-created . . ." as an unanalyzable whole whose meaning is given by (G), then the analogy with (N) does not yield such a startling conclusion. If we give the concept "god" a content along the lines suggested above for "number," then "god" is a sortal under which the objects named by expressions of the form "the-god-which-created x" fall—in which case gods are just worlds, not theological persons.

There is still some residual disanalogy between G-objects (or worlds) and numbers, which may be thought to lend Field some support. After all, (N) is supposed to support a claim for the necessary existence of numbers, whereas one would surely not wish to claim the necessary existence of worlds. But this disanalogy also reflects a difference between Field's version of the argument and Wright's original. Numerical existence is founded on the truth of "F is equinumerous with F "and the existence of an appropriate concept F (e.g., "not self-identical" for 0). In the case of (G), however, the existence of a world is similarly dependent on not only the truth of "x is spatio-temporally related to x," but also on the existence of a spatio-temporal object x. If the existence of the latter is not necessary, then neither is the existence of a world.

The question is then raised, do there exist, necessarily, appropriate concepts within the scope of the concept variables F and G in (N)? There are two issues here: what, in this context, is meant by a concept, and when is a concept appropriate?

Inappropriate concepts are those among which it is not possible to establish 1-1 correlations, such as stuff concepts or concepts whose application is vague. To ask for the number of the concept "water" appears oddly illegitimate. (N) explains this on the grounds that it is not possible to set up relations between instances of the concept "water" and other things. This, in turn, is explained by the ambiguity in the notion of an instance of the concept "water." Quine seeks to resolve this ambiguity by stipulating water to have as its instance all the water there is, taken as one entity, in which case, the number of the concept "water" is one. Alternatively, we may take disambiguation to be achieved only by the addition of a second concept, such as "bucket of" or "molecule of," in which case we can now ask questions concerning number. What is the number of buckets of water? The question, How many books are there on the shelf? may have no answer because vagueness in the term "book" leaves it undetermined whether my address book, copies of *Dialogue*, and so on, are genuine books, and therefore undetermined whether any relation of 1-1 correlation between books on my shelf and other things should include my address book and copies of journals among its relata. Thus, certain concepts cannot figure in 1-1 correlations, even with themselves. Consequently, it is the case that (N) does not yield a number for every concept. But there is no reason to suppose that it cannot for some. The logicist claims that it does.

The remaining issue—What, in this context, is meant by a concept?—raises notorious problems. On the one hand, a psychological notion of concept will not do, since that would make the existence of numbers dependent on the existence of psychological entities. On the other hand, the Fregean notion of a concept as the sort of thing which may be the reference of a predicate tends to lead us toward paradoxes of "the concept

horse" variety. A definitive answer, however, is not required for the purposes of demonstrating the necessary existence of numbers. All we need is a necessarily true proposition which can be regarded as an instance of the right-hand side of (N). As long as the right-hand side is necessary, it is not essential to be able to say exactly what sort of things are the referents of the expressions in place of F and G. The right-hand side of (N) says F and G are 1-1 correlated. Spelt out, this is:

(14) $\exists R\{\forall x\,(Fx \rightarrow \exists y(Gx \,\&\, Rxy)) \,\&\, \forall y(Gy \rightarrow \exists x(Fx \,\&\, Rxy))$
$\&\; \forall x\forall y\forall z[Rxy \,\&\, Rzy \rightarrow (Fx \rightarrow (Fz \rightarrow (Gy \rightarrow x{=}z)))$
$\&\; Rxy \,\&\, Rxz \rightarrow (Fx \rightarrow (Gy \rightarrow (Gz \rightarrow y{=}z)))]\}$

If we let Fx and Gx be $x{\neq}x$, and substitute $x{=}y$ for occurrences of Rxy, etc., then we have:

(15) $\{\forall x\,(x{\neq}x \rightarrow \exists y(x{\neq}x \,\&\, x{=}y)) \,\&\, \forall y(y{\neq}y \rightarrow \exists x(Fx \,\&\, x{=}y))$
$\&\; \forall x\forall y\forall z[x{=}y \,\&\, z{=}y \rightarrow (x{\neq}x \rightarrow (z{\neq}z \rightarrow (y{\neq}y \rightarrow x{=}z)))$
$\&\; x{=}y \,\&\, x{=}z \rightarrow (x{\neq}x \rightarrow (y{\neq}y \rightarrow (z{\neq}z \rightarrow y{=}z)))]\}$

(15) is a logical truth, from which it follows that (14) is a logical truth, where Fx and Gx are understood as standing for $x{\neq}x$ (and similarly Fy, Gy, Fz, Gz). As argued above, (N) then yields as a logical truth the existence of a number, which is the number of xs such that $x{\neq}x$; more usually referred to by the numeral "0." To return to the contrast with (G), it seems simply false to think that there may be logical truths which are instances of the right-hand side of (G), for that would require the existence of some spatio-temporal entity to be a logical truth; but we do have logical truths which are instances of the right-hand side of (N).

6.

I have claimed that Hume's principle is a conceptual truth. It is indeed *true* (in whatever sense *conceptual* truths are true), not a possibly false theoretical claim. It is a conceptual truth in that it is constitutive of the concept of number. But it would be a mistake to think that this fact deprived the numerical expressions thus introduced of any ability to refer, for that requires a dichotomy between those parts of a language which are ontologically committed and those which are not. This dichotomy cannot be sustained because there is not the stock of logically independent elementary propositions that this would require. Being a conceptual truth, Hume's principle is thus part of what I have called broad logic. Thus, arithmetic, including its commitment to the existence of numbers, must be regarded as logical.[13]

Notes

1 See Crispin Wright, *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press, 1983), and George Boolos, "The Consistency of Frege's Foundations of Arithmetic," in *On Being and Saying: Essays for Richard Cartwright*, edited by Judith Jarvis Thomson (Cambridge, MA: MIT Press, 1987), pp. 3-20.

2 I suspect that questions regarding the nature of logic have been largely stultified by the formal success of symbolic logic. For instance, Kneale takes as a condition that a system of logic should be complete, on the grounds that lack of completeness would demonstrate a failure to formalize the basic concepts of the system.

3 I regard the characterization of broad logic as logic to be a principled characterization, whereas to take logic to be narrow logic lacks a strong philosophical justification. Susan Haack, for instance, in discussing the scope of (narrow) logic, says that her "feeling is that the prospects for a well-motivated criterion are not very promising" (*Philosophy of Logics* [Cambridge: Cambridge University Press, 1978], p.7). While sympathetic to Ryle's notion that logic is "topic-neutral," she doubts that it can be made very precise.

4 I am taking narrow logic to include second-order logic. But as explained, nothing depends on this, and we could talk in terms of broad logic throughout.

5 Wright expresses this in terms of the syntactic priority thesis: syntactic categories determine ontological categories. In particular, the category of objects is determined by the the category of singular terms. Here, "determines" means that there is no more to reference to the appropriate entity than use of the appropriate syntactic expression in a true sentence.

6 This is similar to the approach employed by Hilbert and Bernays in *Grundlagen der Mathematik I* (Berlin: Springer, 1934). They allow the expression $\iota_X \Im(x)$ to be a term only if the corresponding *Unitätsformeln* (unity-formulae) have been proved: $\exists x \Im(x)$ and $\forall x \forall y(\Im(x) \,\&\, \Im(y) \rightarrow x{=}y)$. Of course, such an explanation of the syntax of the language leaves open the possibility that its syntax is undecidable. While this is for certain purposes an inconvenience in a formal system, there is no reason to suppose that it cannot be a feature of those parts of natural languages we are trying to explicate. On the contrary, it is a natural reflection of English usage that we are not prepared to accept the introduction of a definite description unless there are grounds for thinking that there is a unique object with the property in question. This approach has the advantage over Russell's theory of descriptions that it allows definite descriptions to be genuine singular terms rather than incomplete symbols. Frege allows definite descriptions always to be named by the highly artificial device of making its denotation be the course of values of the corresponding property, except when the course of values has a unique element, in which case the denotation is that element.

7 Predicates appending to numerical singular terms may be introduced thus: let "#F" symbolize "the number of Fs" and "F and G are 1-1 correlated by R" be symbolized by "$F{\approx_R}G$" so that Hume's principle becomes: "$\#F{=}\#G \leftrightarrow F{\approx_R}G$." Then

for each statement of the form ϕ (#G) there is some F such that ϕ (#G) iff F(G), where F is some property of concepts for which 1-1 correlation is a congruence relation (e.g., #G is even iff $\exists H\{\forall x(Hx \rightarrow Gx)$ & $\exists R$ $H \approx_R (G \& \neg H)\}$. We can see that 1-1 correlation is a congruence relation for this property. Let M be a 1-1 correlation between G and P. We want to show that if $\exists H\{\forall x(Hx \rightarrow Gx)$ & $\exists R$ $H \approx_R (G \& \neg H)\}$ then $\exists Q\{\forall x(Qx \rightarrow Px)$ & $\exists T$ $Q \approx_T (P \& \neg Q)\}$. The property Q we can find and define thus: **Qt** iff **t** is the correlate of some **u** under the correlation M, and **Hu** (i.e., {x:Qx} is the image under M of {x:Hx}). Then, G and P are 1-1 correlated by M, H and Q are 1-1 correlated by M; therefore (G&¬H) and (P&¬Q) are 1-1 correlated by M. Ex hypothesi H and (G&¬H) are 1-1 correlated; since Q and H are 1-1 correlated, and (G&¬H) and (P&¬Q) are 1-1 correlated, it follows that Q and (P&¬Q) are 1-1 correlated, which was what remained to be proved.) Existential quantification over numbers will then be explained thus: $\exists x \phi$ x iff $\exists GF(G)$, where F corresponds to ϕ as above. For details of the derivation of Peano's axioms, see Wright, *Frege's Conception* pp. 154-69.

8 Note that the reductionist cannot simply claim that Hume's principle amounts to an eliminative definition, for it does not. If we define the individual numbers as we have done, we will find embedded occurrences of the numerical operator# (the number of . . .), e.g., $1 = \#(x=(\#(x \neq x)))$ and $2 = \#(x=\#(x \neq x) \vee x=\#(x=(\#(x \neq x))))$. These occurrences cannot generally be eliminated, even though Hume's principle, with the derivative conditions supplied in note 7, supplies truth conditions for statements containing these expressions.

9 Bob Hale, *Abstract Objects* (Oxford: Blackwell, 1987), pp. 4, 7.

10 Boolos, "The Consistency of Frege's Foundations of Arithmetic," p. 18.

11 Hartry Field, "The Conceptual Contingency of Mathematical Objects," *Mind*, 102 (April 1993): 285-99.

12 An equivalence of the form of (I) does not alone determine a unique operator **s** beyond isomorphism. Whether this constitutes an objection to the claim that such equivalences can introduce genuine singular terms (referring expressions) is an issue which goes beyond the scope of this paper. It may nonetheless be pertinent to consider that Quinean arguments suggest that, in any case, no practice can determine reference beyond isomorphism; for a defence of (N) in particular against accusations of indeterminacy, see Bob Hale, *Abstract Objects*, pp. 194-244. Nothing I have said prevents filling out the concept of the kinds of object introduced by instances of (I); so that in the case of (N) it may become clear that numbers are a certain sort of class. It would still remain the case that questions of the identity of such objects would have either to be parasitic upon or, at least, cohere with any possibility of settling such questions by reference to equinumerosity of associated concepts (or by reference to the appropriate R for other instances of (I)). That is what makes them the sort of object they are—i.e., numbers for (N). And so such additional conceptual detail would not detract from the ontological claim made here, that equivalences such as (I) can be conceptual truths which govern the use of genuinely referring singular terms. In the parochial context of Field's argument, the issue

is not whether such reference is determinate, but whether, were there determinate reference, the objects thereby referred to would be objects whose existence ought not to be decidable by the logicist's argument.

13 I am grateful to Professor Hugh Mellor and the anonymous referee for their helpful comments on an earlier draft of this paper.