# PEANO AND RUSSELL

## Introduction

Gottlob Frege is customarily awarded the honorific title 'the father of modern logic'. Jean van Heijenoort (1992: 242) tells us that 'Modern logic began in 1879, the year in which Gottlob Frege (1848–1925) published his *Begriffsschrift*', while Michael Dummett (1993: xiii) regards Frege as the 'initiator of the modern period in the study of logic'. Yet, the praise so frequently accorded to the originality and quality of Frege's logic and philosophy has obscured the significance and influence of his contemporary, Guiseppe Peano. The contribution of the latter has been underestimated; indeed in terms of causal consequence, Peano's work had a greater impact on the development of logic at a crucial period surrounding the turn of the nineteenth century. Peano's influence was felt principally through his impact on Bertrand Russell (indeed, much the same can be said of Frege). In this essay we shall examine this line of influence, looking first at Peano's logic and then at Russell's.

## Guiseppe Peano

Guiseppe Peano was born in Piedmont, in 1858, ten years after Frege and fourteen years before Russell. He was educated in the Piedmontese capital, Turin, where he became professor of mathematics, teaching and researching in analysis and logic. As with several other mathematicians, his interest in logic was in part motivated by the need to promote rigour in mathematics. The emphasis he places on rigour can be appreciated through a debate between Peano and his colleague Corrado Segre. Segre had made important contributions to algebraic geometry, and with Peano made Turin a notable centre for geometry. Nonetheless, they had quite different at-

titudes towards rigour, proof, and intuition. Segre felt that in order to make progress in a new area of mathematics, it would often be necessary to resort to less rigorous, incomplete procedures (Borga and Palladino 1992: 23). Peano (1891: 66) responded that, 'a result cannot be considered as obtained if it is not rigourously proved, even if exceptions are not known.'

The difference between the intuitive approach and the rigorous approach is apparent in the mathematical understanding of continuity. The *intuitive* notion of a continuous curve is one that can be divided into parts each of which is continuously differentiable. Curves of the latter sort cannot fill a space—however small a space we choose, the curve cannot go through every point in the space. Intuitively a one-dimensional thing, a continuous curve, cannot fill two-dimensional space. However, Peano shows that some continuous curves can go through every point in a finite space, indeed in infinite space. In a paper published in 1890, Peano gave a formal description of what became known as the 'Peano curve', a curve that passes through every point in the unit square. The paper gave no illustration of the curve, precisely because, Peano held, it is thinking in visual terms about geometry and analysis that leads to the use of intuition in mathematics and the breakdown of rigour.

The other influence on the Peano curve was Cantor. Peano was intrigued by Cantor's proof that the set of points in the unit interval (the real numbers between 0 and 1) has the same cardinality as (i.e. can be put into 1–1 correlation with) the number of points in the unit square. Since the cardinality of the unit interval is equal to that of the whole real line, this raises the possibility that there is a curve that proves Cantor's conclusion by going though every point in the unit square. Peano's curve does exactly that.

## Peano's logical symbolism

In the year following publication of his space-filling curve, Peano instituted his 'Formulario' project, the aim of which was to articulate all the theorems of mathematics using a new notation that Peano had devised. A little earlier Peano (1888) has

published a book on the geometrical calculus, following the approach of Hermann Grassmann's *Ausdehnungslehre*. According to Grassmann, geometry does not have physical space as its subject matter. It is rather a purely abstract calculus that may have application to physical space. Conceiving it this way allows for a formal approach to geometrical proof divorced from intuition and thereby allows for flaws in intuition-based reasoning (for example in Euclid) to be avoided. Moritz Pasch, in his *Vorlesungen über neuere Geometrie* (1882) takes certain terms as primitive and undefined ('Kernbegriffe' as Pasch later called them) while certain theorems ('Kernsätze') are taken as unproved, i.e. are the axioms. Derived theorems are to be proved by pure logic alone, independently of any intuition. Pasch's approach influenced Peano's *I principii di geometria logicamente esposti* (1889). For this programme to be realised fully, Peano held, the geometrical calculus needed the framework of a logical calculus.

In 1889 Peano also published his *Arithmetices principia, nova methodo exposita*. This work (in Latin) contained his famous five postulates for arithmetic—the 'arithmetices principia'. The phrase 'nova methodo exposita' refers to the new symbolism that Peano introduced for logical relations: the now familiar '$\epsilon$' for set/class membership and 'Ɔ' for implication (a rotated 'C' for 'consequentia'—in the second volume of the *Formulaire* Peano replaced 'Ɔ' by the now-familiar horseshoe'$\supset$'). Peano has already used several other new symbols that were to become central to the symbolism of logic and set-theory, such as the symbols for union and intersection, '$\cup$' and '$\cap$', and existential quantification, '$\exists$', and the tilde '$\sim$' for negation (Cajori 1929: 298–302). Peano signified universal generalisation using subscripts with his 'Ɔ'. The familiarity of these symbols tends to mask the importance of their introduction. But the idea is familiar that having the appropriate terminology is crucial not just to expressing thoughts clearly but often also to having the relevant thoughts at all. The development of a symbolism for logic was not just a lubricant to the accelerating vehicle that was mathematical logic, but was an essential part of its motor, as Bertrand Russell was soon to acknowledge.

## From Peano to Russell

In the last summer of the nineteenth century, the first International Congress of Philosophy took place in Paris (it was to be followed immediately by the second International Congress of Mathematics). Peano was on the committee for the Congress and one of its leading lights. Also attending was twenty-eight year old Bertrand Russell. Born into the aristocracy, the grandson of a Prime Minister, Russell came from quite a different background from Peano. Nonetheless, their interests converged on the intersection of mathematics, logic, and philosophy. Russell went up to Cambridge in 1890, where he completed both the Mathematical and Moral Sciences (Philosophy) Triposes. Like Peano, Russell was much taken by the work of Georg Cantor. He had begin to study Cantor's work in 1896 (Grattan–Guinness 1978: 129), and had been thinking a great deal when he met Peano in August 1900. In his autobiography, Russell (1967: 144) records the significance of his meeting thus:

> The Congress was the turning point of my intellectual life, because there I met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument on which he embarked. As the days went by, I decided that this must be owing to his mathematical logic . . . It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years . . .

and thus (Russell 1959: 65):

> I went to [Peano] and said, 'I wish to read all your works. Have you got copies with you?' He had, and I immediately read them all. It was they that gave the impetus to my own views on the principles of mathematics.

## Burali-Forti's paradox and Cantor's theorem

Russell, who had been intending to stay for both congresses, left Paris in order to return home to study Peano's work in greater detail. He was, however, already aware of Burali-Forti's paradox and of Cantor's paradox. Cesare Burali-Forti was Peano's assistant and lectured on Peano's mathematical logic and used Peano's symbolism in his own work. In 1897 Burali-Forti had published his famous paradox concerning ordinals. In a simple terms the class of all ordinals, Ω, is an ordinal or corresponds to an ordinal. But then we can construct a greater ordinal, which is both a member of Ω, (because it is an ordinal) and not a member of Ω, (because it is greater than Ω). Georg Cantor had earlier (1891) published his diagonal proof that the power set of a set is always larger than the set itself (as follows).

Let us assume that there is a 1-1 correspondence between $S$ and its power set, $\mathscr{P}(S)$. This 1-1 correspondence, $f : S \rightarrow \mathscr{P}(S)$ is a bijection; for every member of $x$ of S, $f(x) \in \mathscr{P}$(S), and for every member $y$ of $\mathscr{P}$(S), there is a unique $x \in$ S such that $f(x) = y$ (Table 1).

| $S$ | $\mathscr{P}(S)$ |
|---|---|
| $x$ | $f(x)$ |

Table 1: $f$ is a 1-1 correlation between S and $\mathscr{P}$(S)

The various $f(x)$ are subsets of $S$. And so some members of S may be correlated by $f$ with sets of which they are themselves members, i.e. $x \in f(x)$. Let us use 'Σ' to denote the set of such members of S, i.e. $\Sigma = \{x : x \in S \wedge x \in f(x)\}$. Now consider the remaining members of S; call this set 'Σ*'. Σ* contains those members of S which are not members of the sets with which they correlate under f; i.e. $\Sigma^* = \{x : x \in S \wedge x \notin f(x)\}$ (Table 2).

Every member of $S$ is a member of one or other of Σ and Σ* but not of both. Both Σ and Σ* are subsets of $S$, hence they are members of $\mathscr{P}$(S). This means, by the definition of $f$, there must be members of $S$ which correlate with them. Let '$x^*$' denote the member that correlates with Σ* (i.e. $f(x^*) = \Sigma^*$) (Table 3).

| $S$ | $\mathscr{P}(S)$ | |
|---|---|---|
| $x$ | $f(x)$ | where x $\in f(x)$—such sets are members of $\Sigma$ |
| $z$ | $f(z)$ | where $z \notin f(z)$—such sets are members of $\Sigma^*$ |

Table 2: Some members of $S$ correlate with sets they are members of; these are members of $\Sigma$. Others correlate with sets they are not members of; these are members of $\Sigma^*$.

| $S$ | $\mathscr{P}(S)$ |
|---|---|
| $x^*$ | $\Sigma^*$ |

Table 3: $x^*$ correlates with $\Sigma^* = f(x^*)$

Now we ask: Is $x^*$ a member of $\Sigma$ or $\Sigma^*$? If x* $\notin \Sigma^*$ i.e. $x^* \in \Sigma$, then by definition of $\Sigma$, $x^* \in f(x^*)$ (see Table 2). But by definition of $x^*$ (see Table 3), $f(x^*) = \Sigma^*$, so $x^* \in \Sigma^*$. So we have shown that x* $\notin \Sigma^* \rightarrow$ x* $\in \Sigma^*$.

If on the other hand $x^* \in \Sigma^*$, then by definition of $\Sigma^*$ (see Table 2), we have $x^* \notin f(x^*)$. Since $f(x^*) = \Sigma^*$ (see Table 3), it follows that $x^* \notin \Sigma^*$. That is, we have shown that $x^* \in \Sigma^* \rightarrow x^* \notin \Sigma^*$.

So together we have: $x^* \in \Sigma^* \leftrightarrow x^* \notin \Sigma^*$, which is a contradiction. Thus the assumption that there is a 1-1 correlation between S and $\mathscr{P}$(S) is false.

But Russell had not initially seen the full significance of this result. Russell (1963: 69) wrote: '…in this one point the master [Cantor] has been guilty of a very subtle fallacy, which I hope to explain in some future work.' Nonetheless, in due course, most probably as a result of thinking about Burali-Forti's paradox and being assisted by Peano's symbolism, he came to see the implications of Cantor's proof and of Burali-Forti's paradox. Cantor himself, had, around the same time as Burali-Forti, drawn the conclusion from his theorem, that there is no largest cardinal. Russell later remarked, 'I attempted to discover some flaw in Cantor's proof that there is no largest cardinal …Applying this proof to the supposed class of all imaginable ob-

jects, I was led to a new and simpler contradiction . . . '. The contradiction to which he was led was that of the set of sets which do not contain themselves.

## Russell and Frege

In 1903 Russell published his *Principles of Mathematics*, in which he set out the logicist programme he was to pursue with Alfred North Whitehead. Although Russell had already been at work on a book with this title, he was impelled to rewrite the work completely as a result of meeting Peano, which he did in the last three months of 1900 (Russell 1959: 72–3). At this time Russell was not aware of Frege's work. Indeed Russell came across Frege by reading the works of Peano, which Peano had presented to him in Paris. On seeing Frege's *Grundgesetze der Arithmetik* (Basic Laws of Arithmetic) and *Grundlagen der Arithmetik* (Foundations of Arithmetic), Russell was struck by the philosophical significance of the latter and by the fatal flaw in the former. Because Russell engaged with Frege's work only once he had completed writing the *Principles of Mathematics*, Russell chose, in the light of its philosophical significance to add an extended appendix in which Frege's work is discussed. Russell says:

> The work of Frege, which appears to be far, less known than it deserves, contains many of the doctrines set forth in Parts I and II of the present work, and where it differs from the views which I have advocated, the differences demand discussion. Frege's work abounds in subtle distinctions, and avoids all the usual fallacies which beset writers on Logic. His symbolism, though unfortunately so cumbrous as to be very difficult to employ in practice, is based upon an analysis of logical notions much more profound than Peano's, and is philosophically very superior to its more convenient rival. In what follows, I shall try briefly to expound Frege's theories on the most important points, and to explain my grounds for differing where I do differ. But the points of disagreement are very few and slight compared to those of agreement.

The conclusion we should draw, I believe, is that despite the quality of Frege's work, it was not especially influential in the early development of modern logic. Russell is the crucial figure (perhaps with Hilbert alongside him). But although Frege's work may have sharpened Russell's subsequent philosophical thinking, there is no reason to suppose that Russell's logic would have been much different without Frege. Even if less worthy, Peano's logic was, as we have seen, seminal for Russell.

As for the fatal flaw, Russell immediately saw that Frege's axioms in his Basic Laws led to a contradiction. As Russell wrote to Frege, just as the completion of the latter was in print, (June 16, 1902, letter to Frege, translation in (van Heijenoort 1967: 125)):

> Let $w$ be the predicate: to be a predicate that cannot be predicated of itself. Can $w$ be predicated of itself? From each answer its opposite follows. Therefore we must conclude that $w$ is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection does not form a totality.[1]

Frege's fifth basic law, (V), claims that every predicate, F, defines a set, its extension, which I symbolise thus '$\hat{}$F' in the following. Russell's $w$ seems to be precisely a predicate that does not have an extension.

Let F$x$ be $x \notin x$ (def F). (Def $\in$: $x \in y \leftrightarrow \exists$ G($y = \hat{}$G $\wedge$ G$x$).) By (V) we have an extension for F, *viz.*$\hat{}$F. We now consider whether or not F($\hat{}$F).

| | | |
|---|---|---|
| (1) | ¬F($\hat{}$F) | assumption |
| (2) | $\hat{}$F$\in\hat{}$F | (1, def F, double negation) |
| (3) | $\exists$ G($\hat{}$F=$\hat{}$G $\wedge$ G($\hat{}$F)) | (2, def $\in$) |
| (4) | $\hat{}$F=$\hat{}$G $\rightarrow$ (F$\leftrightarrow$G) | (V) |
| (5) | $\exists$G(F$\leftrightarrow$G$\wedge$ G($\hat{}$F)) | (3, 4 MP) |
| (6) | F($\hat{}$F) | (5, simple logic) |

[1] It is worth remarking that in this letter Russell expresses the contradiction explicitly using Peano's notation, thus: $w$ = cls $\cap$ $x$ $\ni$($x \sim\varepsilon x$).$\supset$: $w \varepsilon w$ .=. $w \sim\varepsilon w$. Russell says that he had written to Peano about this.

(7) $\hat{}F \in \hat{}F$ (from def $\in$)

(8) $\neg F(\hat{}F)$ (def F)

And so considering lines 1, 6 and 8, we have: $\neg F(\hat{}F) \leftrightarrow F(\hat{}F)$, which is Russell's paradox.

Another of the paradoxes considered by Russell was that discovered by the French mathematician Jules Richard. Russell gives it as follows. Consider all decimals that can be defined using of a finite number of words; let E be the class of all such decimals. Since the possible finite expressions of a natural language are countable, the number of members of E is countable. Hence it is possible to list the members of E; one decimal in E will be the first in the list, another the second, another the third, and so on. We define the decimal N as follows: if the *n*th digit in the *n*th decimal is *p*, let the *n*th digit in N be *p*+1, or 0 if *p*=9. Then N is different from all the members of E, since whatever value *n* may have, the *n*th digit in N is different from the *n*th digit in the *n*th of the decimals in the list of E (i.e. N is constructed by Cantor's diagonal method). Nevertheless we have defined N in a finite number of words, and therefore N ought to be a member of E. Thus N both is and is not a member of E. (A simpler version of the paradox is this: let M = the least integer not definable in fewer than twenty syllables. M has just been defined in nineteen syllables, so M both is and is not definable in fewer than twenty syllables.)

The paradox is dissolved, according to Richard, if we hold that the definition of E, as the totality consisting of all decimals that can be finitely defined, cannot be regarded as also ranging over decimals which are defined by reference to that totality. Thus given that N is defined that way, it is not a legitimate possibility that N is in E.

## The vicious circle principle

Richard's dissolution exemplifies the vicious circle principle that was used by Russell and by Henri Poincaré in order to exclude viciously circular definition.

Russell states the principle thus: '"Whatever involves all of a collection must not be one of the collection", or, conversely: "If, provided a certain collection had a total,

it would have members only definable in terms of that total, then the said collection has no total".' And in a footnote he explains this last: 'When I say that a collection has no total, I mean that statements about all its members are nonsense.' Although the application of the principle is, as it stands, not entirely clear, it is probably overstating matters to claim, as Kurt Gödel did, that Russell gives us not one but three principles. There is just one one principle: if a definition is to be meaningful then if the definiens refers to all of some totality the definiendum may not be part of that totality.

The aim of the vicious circle principle is to exclude impredicative definitions. The idea of impredicativity, never made entirely clear, was roughly this: a definition is not impredicative if the definiendum is defined in terms of predicates to which it is not a possible argument. I.e. the definiendum is defined in terms of totalities (the argument domains of the predicates) of which it is not a part. In impredicative definitions, the definiendum is a possible argument of the predicate and is included in the totality according to which it itself is defined.

Let us see how this works for contradictions resulting from unrestricted comprehension (essentially the same as the contradiction emanating from Frege's axiom (V)). Unrestricted comprehension says: for any predicate $\phi$ there is a class $y$ of objects satisfying $\phi$, i.e. for any $\phi$, $\exists y \forall x (x \in y \leftrightarrow \phi x)$. The contradiction arises from taking $\phi x$ to be $x \notin x$, i.e. there is a $y$ such that $\forall x (x \in y \leftrightarrow x \notin x)$. Substituting $y$ for $x$ (since $x$ ranges over all classes) we have $y \in y \leftrightarrow y \notin y$. This definition of $y$ is impredicative since $y$ is being defined in terms of a predicate, $x \in x$, for which $y$ is itself a possible argument. Russell takes the vicious circle principle to exclude such definitions.

The vicious circle principle is negative, pointing to one source of contradiction which must be excluded. It is intended as a criterion of meaningfulness; we must construct our mathematical logic and its language in such a way that expressions that violate the principle are meaningless. On which positive principles the logic should be constructed the vicious circle principle does not directly suggest. However, different interpretations of why violation leads to contradiction will encourage different approaches to a positive logic. Poincaré, on the one hand, saw the illegiti-

macy in talking of all of the class of finitely definable decimals as deriving from the fact that that class is infinitely large. Hence logic must be constructed in such a way as to allow quantification only over finite classes.

## Russell's theory of types

Russell, on the other hand, saw the issue as turning on logical homogeneity. The things we attempt to talk about may not be all of the same logical kind or type. According to Russell, it should be improper to quantify over things of a different logical type. If one assumes that things which involve all of a collection are different sorts of thing from the members of the collection, then the requirement of logical homogeneity would forbid quantifying over members of a collection and the collection itself at the same time. The contradictions arise because the heterogeneity, the difference is kind, is hidden by the description given of the object. The hidden heterogeneity of the collections appears to license including in the collection the very entity which is defined in terms of the collection. Consequently there would be no violations of the vicious circle principle in a logic which precludes talking of heterogeneous collections, but allows quantification only over entities of the same logical type.

This is precisely what Whitehead and Russell's theory of types in *Principia Mathematica* aims to do. Alfred North Whitehead, who had taught Russell mathematics at Trinity College, Cambridge, and Russell colaborated on this work with two goals. The first was to achieve that logcist programme that Frege had inspired, showing that the truths of arithmetic can be deduced from those of logic. The second was to construct an underlying logic that avoids the paradoxes in a principled way, and this is the contribution of the theory of types. How it does this is again most easily seen in connection with the abstraction axiom (naive/unrestricted comprehension): for all $\phi$, $\exists y \forall x (x \in y \leftrightarrow \phi x)$. We have seen how this axiom leads to contradiction. The theory of types, by submitting to the vicious circle principle, avoids the contradiction. Classes are not allowed to be members of themselves or of classes of the same type as themselves; formulae such as $x \in x$ and $x \notin x$ are simply ill-formed, they are

not formulae of the language and hence the question of their truth or falsity does not arise. Thus they do not represent predicates that can take the place of $\phi$ in the above abstraction axiom. More generally our stricture prevents the universal and existential quantifiers in the axiom, the '$\forall x$' and '$\exists y$' in $\exists y \forall x (x \in y \leftrightarrow \phi x)$, from overlapping. If their ranges did overlap, as they are allowed to do in the derivation of the contradiction, we would be forced to contemplate illegitimate possibilities such as $y \in y$. The quantifiers are restricted each to one homogeneous type.

These logical types form a hierarchy, at the very bottom of which comes the type of individuals, which aren't really classes at all. The next level up, the first type of classes, contains only those classes which contain individuals as members. The next higher type is the type of classes whose members are the classes of the first type, i.e. the type of classes of classes of individuals. The hierarchy continues in such a way that each class is assigned a particular type, and if this type is the $n+1$th, its elements are of the $n$th type, and it itself may only be a member of those classes of the type $n+2$.

Correspondingly, we allow it to make sense to assert that one set is a member of another only if the type of the latter is one higher than that of the former. Hence quantification must be only within a single type; to be exact, in a formula such as '$\forall x F x$' we should specify which type of classes the variable $x$ ranges over. If this is specified as $n$, then the gloss for '$\forall x \mathrm{F} x$' starts 'for all classes of the $n$th type ...' which we might emphasize by writing: '$\forall x_n \mathrm{F} x_n$'. Our axiom of comprehension then looks like this: $\exists y_{n+1} \forall x_n (x_\mathrm{n} \in y_{n+1} \leftrightarrow \phi x_n)$.

It should be noted that our hierarchy means that it makes no sense to talk of 'class' (nor of 'cardinal' or 'ordinal' etc.) simpliciter, for the term is ambiguous, not referring to one sort of object, but many sorts. Thus use of the term 'class' must be governed by a systematic ambiguity, that is, taken as referring to classes of some one, albeit unspecified, type.

This hierarchy of classes does a nice job of excluding the set theoretic or class paradoxes: Russell's, Cantor's, Burali-Forti's. It seemed as if classes were all of one sort and that we could quantify over all of them. In *Principia Mathematica* it turns

out that classes may be rather different things and that we cannot quantify over all with one variable.

## Dissolving the semantic paradoxes

On the other hand, the paradox from which we first gleaned our guiding vicious circle principle, the Richard Paradox, and those like it appear to be somewhat different. In the class case we applied the principle that a class which is defined in terms of some totality of classes is a different sort of class from those in that totality, hence it could not be part of that totality. But in the case of the Richard paradox we tried to define one decimal, N, in terms of a particular class, E, of decimals. In this case it could not possibly be said that N is a different sort of object from the members of E. On the contrary, N is a decimal just like the members of E, and assuming the paradox is dissolved in such a way that N does exist, there may well be classes containing N and the members of E.

It would appear that the semantic paradoxes are not to be dissolved by pointing to hierarchies of entities which differ in type. Let us return to the vicious circle principle. It warned us not to refer to illegitimate totalities. Russell's approach was then to seek in each paradox the illegitimate totality referred to. In his own paradox it was 'all classes', in the Burali-Forti contradiction it was 'all ordinals' etc. In the case of the Richard paradox our problem is not the totality 'all decimals' but 'all definitions of decimals', for the crucial question was whether the *definition* of N belongs to this totality. Similarly the paradox of the least indefinable ordinal refers to all definitions. Berry's paradox refers to all names. What we must then do is to seek a hierarchy of definitions or names (which will amount to the same thing since the definitions and names are both given by description, that is, in terms of properties). And indeed, this part of Whitehead and Russell's theory, the theory of orders, proceeds precisely analogously to the theory of class-types. If a property (strictly, a propositional function) is defined in terms of some totality of properties, then the defined property is of a different sort, i.e. of a higher order. The term 'property' is itself ambiguous,

requiring specification, at least in principle, of some specific order. Variables range only over properties of one order.

## The no-class theory

Russell thought that if classes really did exist as objects independent of functions or of constructions then there would be no logical reason why a class might not be a member of itself. More significantly, Russell held, like Frege, that there could be no restriction on variables preventing them from ranging over all objects in the world that would not be arbitrary. Thus if classes were real, there could be no objection to our talking of the totality of classes which do not contain themselves and other such totalities which lead to the contradictions. Thus the theory of types requires taking a non-platonist, non-realist view of classes

By making classes derivative of functions, the hierarchy of classes could be justified on the ground of the distinction between functions. The justification is most natural if functions are seen as linguistic entities constructed in terms of other functions already defined. Clearly such a view obeys the vicious circle principle as it was formulated above; contravention would be a case of circular definition, obviously giving no meaning to the sign in question. Furthermore, it coheres with Russell's logical atomism. Nonetheless there is nothing in this to prevent a function taking arguments of more than one type so long as all such types are lower than the type of the function itself, as in a cumulative hierarchy like Zermelo–Frankel set theory. In *Principia Mathematica* this possibility is excluded.

In fact Russell and Whitehead come later, in the second edition of *Principia Mathematica* (1925–7), to justify certain contentious axioms (the axioms of reducibility and infinity) abductively; that is, by virtue of their ability to generate 'true' theorems. This marks a departure from their original logicist programme of founding mathematics on evident logical axioms.

## Problems with the theory of types

The theory of types successfully avoids the sort of contradictions faced by Frege's logic. However it does depend on two axioms which do not seem to be true logical principles. One is the axiom of reducibility and the other is the axiom of infinity. The former need not concern us here. The need for the axiom of infinity arises because we are not able to quantify over entities belonging to different types. Imagine that there are only a finite number of individuals in the world. Thus only a finite number of entities of type 0. The entities of type 1 will be all the possible classes of entities of type 0. The number of such entities will also be finite ($2^n$ to be precise). Thus there will only ever be a finite number of entities of any given type. This means there will be no type with an infinite number of entities. So we will not be able to get enough classes to represent the natural numbers (which Russell does in Frege's manner: 3 = the class of three membered classes (of a certain type).) Thus we need to have as an axiom, the assertion that there are infinitely many individuals in the universe. This axiom seems to be one which cannot be a truth of logic.

The logicist programme was to receive a further blow with Gödel's 1931 paper 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems', which showed that there are ('undecidable') propositions of arithmetic such that neither they nor their negations can be proven in a system of first-order logic plus Peano's axioms for arithmetic. Furthermore there is no finite, consistent addition to Peano's axioms that is complete. Presumably the undecidable propositions are each either true or false. In which case there are true propositions of arithmetic that cannot be proven in any finite, consistent axiomatization of arithmetic. A fortiori, there is no finite, consistent axiomatization of l(first order) logic that captures all the truths of arithmetic.

## Conclusion

Frege was undoubtedly an important force in the development of modern logic. But his significance can be overstated under the influence of the retrospective appreciation of the quality of both his logical and philosophical work. Frege's work was not

widely read in his lifetime. But two thinkers who did read Frege were Peano and, thanks to Peano, Russell. The influence of Peano and Frege on Russell and Whitehead may be summed up in the Preface to *Principia Mathematica* (Whitehead and Russell 1925: viii):

> In the matter of notation, we have as far as possible followed Peano, supplementing his notation, when necessary, by that of Frege or by that of Schröder ... In all questions of logical analysis, our chief debt is to Frege.

## Reading

Wang *From Mathematics to Philosophy* chapter 4

Sainsbury *Russell* chapter 8

Russell 'Mathematical Logic as Based on the Theory of Types' in *Logic and Knowledge*

Russell *The Principles of Mathematics*

Russell *Introduction to Mathematical Philosophy* (and in Benacerraf and Putnam eds Philosophy of Mathematics)

Godel 'Russell's Mathematical Logic' in Benacerraf and Putnam eds Philosophy of Mathematics

Shapiro *Thinking About Mathematics* c. 5.2

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